The electron-positron annihilation cross section used for high precision tests of the Standard Model

Stefan Groote

Institut für Physik der Johannes-Gutenberg-Universität,
Staudinger Weg 7, 55099 Mainz, Germany

Abstract

Quantities like the fine structure constant at the pole of the Z boson and the anomalous magnetic moment of the muon are of profound importance for testing the Standard Model of elementary particle physics. Because these quantities are known with very high precision, deviations between experimental measurements and theory predictions for these quantities open a window for so-called "new physics", i.e. physics beyond the Standard Model. In this seminar new calculation techniques for the most unreliable part of the theory predictions, the hadronic contribution, are formulated, discussed and used for a prediction of these quantities.

Contents

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1 Parameters for a precision test

The running fine structure constant or QED coupling α at the scale of the Z^0 mass and the anomalous magnetic moment a_μ of the muon [1, 2, 3, 4] are two parameters which have gained great deal of interest during the last years. The reason for this interest is that these parameters can be determined experimentally with high precision. Theoretical predictions, therefore, can serve as precision tests of the Standard Model (SM) of elementary particle physics because deviations to the measurement opens windows for the appearence of so-called "New Physics".

The QED coupling

For the QED coupling, an accurate knowledge of $\alpha(M_Z)$ is instrumental in narrowing down the mass window for the last particle of the Standard Model, the Higgs particle, which might have been already detected at the LEP collider at CERN at 115 GeV but which needs to be verified. It is worth to consider this parameter at the pole of the Z^0 because a huge amount of data have been taken there during a rather long period at the LEP I run. One could expect a rather precise experimental measurement at this point.

The anomalous magnetic moment

The above mentioned window seems to open already for the anomalous magnetic moment of the muon. In February 2001, the Brookhaven National Laboratory (BNL) reported about a precision measurement of the anomalous magnetic moment of the positive muon [5],

$$a_\mu^{\rm exp} = (116\,592\,023 \pm 151) \times 10^{-11} \tag{1}$$

which had to be contrasted with the theory prediction. At the first moment, a deviation of roughly 2.6 standard deviations gave rise to suggestions for new-physics effects published during the last few months, including concepts of supersymmetry, leptoquarks, lepton number violating models, technicolor models, string theory concepts, extra dimensions and so on (cf. for instance Ref. [6]). However, the situation changed suddenly when an error in the sign was found in the calculation of the light-by-light contribution, obtained by Tom Kinoshita and collaborators [7] (see e.g. Refs. [8]). This sign was later on corrected by the authors themselves [9]. The discrepancy was reduced again. New data sets taken in Novosibirsk again churn the water, but the physics community is split into two parts at least since these events.

The hadronic contribution

From the theoretical point of view the uncertainty in the determination of $\alpha(M_Z)$ and a_μ is dominated by the uncertainty of the hadronic contribution $\alpha^{\rm had}$ and $a_\mu^{\rm had}$. This talk, therefore, will deal with these contributions only.

2 The integral representation

In this section I want to explain where the hadronic contribution comes from. Actually, the main ingredient to this contribution is the hadronic two-point correlator of the photon current, and the details and reasons of their origin are detailed in the following subsections.

Hadronic contribution to the QED coupling

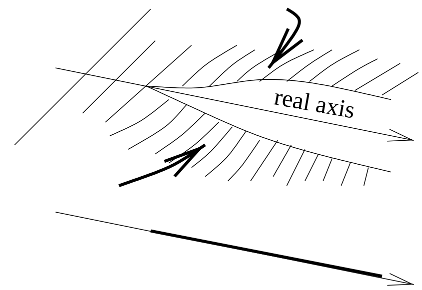
The corrections $\Delta\alpha(q^2)$ to the QED coupling constant are given by the corrections to the photon propagator due to the calculation of a chain of inserted vacuum-polarization terms [1, 10, 11],

$$\begin{aligned} \frac{-\alpha(q^2)}{q^2} &= \alpha\left(\frac{-1}{q^2}+\frac{-1}{q^2}\Pi_\gamma(-q^2)\frac{-1}{q^2}+\dots\right) = \\ &= \frac{-\alpha}{q^2(1+\Pi_\gamma(-q^2)/q^2)} = \frac{-\alpha}{q^2(1-\Delta\alpha(q^2))} \end{aligned} \tag{2}$$

where $\Pi_\gamma(-q^2) = -e^2q^2\Pi(-q^2) = -4\pi\alpha q^2\Pi(-q^2)$ and $\Pi(-q^2)$ is the two-point correlator. This function is an analytical function except for a cut along the real axis for $q^2 < -4m_\pi^2$, and it vanishes for $|q^2| \to \infty$. In order to understand what is meant by this, I take a simple example, namely

$$\Pi_M(q^2) = \sqrt{4m_\pi^2 - q^2}. \tag{3}$$

This function takes an imaginary value for $q^2 > 4m_\pi^2$. The sign of the imaginary value depends on whether we approach the real axis from the upper or lower half plane, as it is shown in the figure on the right hand side. This should be indicated by $q^2 = se^{\pm i0}$, $s > 4m_\pi^2$. So we obtain



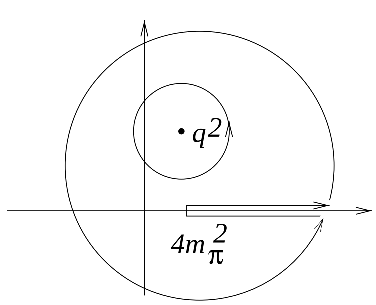
$$\begin{aligned} \sqrt{4m_\pi^2 - se^{i0}} &= \sqrt{4m_\pi^2 + se^{i\pi}} = \sqrt{(s-4m_\pi^2)e^{i\pi}} = \\ &= e^{i\pi/2}\sqrt{s-4m_\pi^2} = i\sqrt{s-4m_\pi^2}, \\ \sqrt{4m_\pi^2 - se^{-i0}} &= e^{-i\pi/2}\sqrt{s-4m_\pi^2} = -i\sqrt{s-4m_\pi^2}. \end{aligned} \tag{4}$$

We define the discontinuity by

$$Disc\,\Pi_M(s) = \Pi_M(se^{i0}) - \Pi_M(se^{-i0}) = 2i\sqrt{s-4m_\pi^2}. \tag{5}$$

The spectral density is finally given by

$$\rho_M(s) = \frac{1}{2\pi i}Disc\,\Pi_M(s) = \frac{1}{\pi}\sqrt{s-4m_\pi^2}. \tag{6}$$



After having investigated the cut by introducing the discontinuity, we can use Cauchy's theorem in order to express the correlator function by the corresponding spectral density. For this we take the circle about the specific point $z = q^2$ and expand this circle to a circle with infinite radius. If the circle reaches the cut, it will circumvent this cut along the real axis, resulting in two line integrals. In assuming that in the limit of an infinite radius the circle part of the integral vanishes,

$$\Pi_M(q^2) = \frac{1}{2\pi i}\oint\frac{\Pi_M(z)dz}{z-q^2} =$$

$$= \frac{1}{2\pi i}\int_{\infty e^{-i0}}^{4m_\pi^2 e^{-i0}} \frac{\Pi_M(z)dz}{z-q^2} + \frac{1}{2\pi i}\int_{4m_\pi^2 e^{i0}}^{\infty e^{i0}} \frac{\Pi_M(z)dz}{z-q^2} =$$
$$= \frac{1}{2\pi i}\int_{4m_\pi^2}^{\infty} \frac{\Pi_M(s^{i0})-\Pi_M(s^{-i0})}{s-q^2}ds = \int_{4m_\pi^2}^{\infty}\frac{\rho_M(s)}{s-q^2}. \tag{7}$$

This identity, known as *dispersion relation*, is valid not only for the used example but generally if $\Pi_M(q^2)$ falls off sufficiently fast for $|q^2| \to 0$. However, this is not the case for the correlator function we have to deal with in our real application,

$$\Pi(q^2) = \int_{4m_\pi^2}^{\infty}\frac{\rho(s)ds}{s+q^2}, \qquad \rho(s) = \frac{1}{2\pi i}Disc\,\Pi(s), \quad Disc\,\Pi(s) = \Pi(se^{-i\pi}) - \Pi(se^{i\pi}) \tag{8}$$

(it is convenient to write the correlator function in Euclidean space). In this case $\Pi(q^2)$ is singular. But we can redefine $\Pi(q^2)$ by a subtracted quantity,

$$\Pi(q^2) \to \Pi(q^2) - \Pi(0) = \int_{4m_\pi^2}^{\infty}\left(\frac{1}{s+q^2} - \frac{1}{q^2}\right)\rho(s)ds = -\int_{4m_\pi^2}^{\infty}\frac{q^2\rho(s)ds}{s(s+q^2)}. \tag{9}$$

This subtraction method is known as *momentum subtraction* at a specified momentum square, in this case at the point $q^2 = 0$. Taking only the hadronic contribution to the correction, the spectral density is related to the relative hadronic cross section in e^+e^- annihilations

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} \tag{10}$$

by $R(s) = 12\pi^2\rho(s)$. For low energies this relative cross section is determined by the pion form factor F_π,

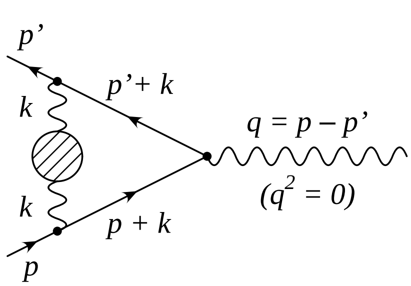
$$R(s) = \frac{v_\pi^3}{4}|F_\pi(s)|^2, \qquad v_\pi = \sqrt{1-\frac{4m_\pi^2}{s}}. \tag{11}$$

So the expression we deal with for the hadronic contribution to the fine structure correction is given by

$$\Delta\alpha(M_Z^2) = -4\pi\alpha\int_{4m_\pi^2}^{\infty}H(s)\rho(s)ds, \qquad H(s) = \frac{M_Z^2}{s(M_Z^2-s)} \tag{12}$$

The integrand consists of the spectral density and a factor which is called *weight function*.

Hadronic contribution to the anomalous magnetic moment



The anomalous magnetic moment can be determined by looking at the first order QED correction to the muon-photon vertex with an self energy insertion for the loop photon. Using again the dispersion relation, we obtain

$$a_\mu = 4\pi^2\left(\frac{\alpha}{\pi}\right)^2\int_{4m_\pi^2}^{\infty}H(s)\rho(s)ds \tag{13}$$

where the weight function is $H(s) = K(s)/s$ with

$$K(s) = \int_0^1 \frac{z^2(1-z)dz}{z^2+(1-z)s/m^2}. \tag{14}$$

Details about how to obtain this function are shown in Appendix A. The integral can be calculated and gives rise to

$$\begin{aligned} K(s) &= \frac{(1+x^2)(1+x)^2}{x^2}\ln(1+x) + \frac{x^2(1+x)}{1-x}\ln(x) + \\ &+\frac{x^2}{2}(2-x^2) + \frac{(1+x^2)(1+x)^2}{x^2}\left(-x+\frac{x^2}{2}\right) \end{aligned} \tag{15}$$

where

$$x = \frac{1-v}{1+v}, \qquad v = \sqrt{1-\frac{4m_\mu^2}{s}}. \tag{16}$$

In this case $K(s)$ has a cut along the real axis so that the total singularity of the weight function $H(s) = K(s)/s$ at the origin is stronger than in the case of the QED coupling. If we again only take the hadronic contribution, $\rho(s)$ is meant to be the spectral density corresponding to the hadronic correlator function.

Local and global duality

The question may arise why we cannot exclude the experimental values from our considerations at all and take the relation in Eq. (8) in order to calculate the spectral density from a correlator function calculated perturbatively. The reason is that there is actually an obstacle in using this relation. As depending on methods of functional analysis, the inverse of the dispersion relation is only valid if there are no poles encircled by the path in the complex plane, a condition which is necessary to obtain this relation. These poles can have their origin from weight functions in combination with the spectral density. This means that if there is such a weight function included in the integration of the spectral density, the inverse relation shown above is only valid *globally* and not *locally*. We call this *global* resp. *local duality*. Two paths out of this situation are shown in the following.

3 The polynomial adjustment method

A singularity in the encirled part of the complex plane occurs for the two parameters we want to look at. Therefore we cannot fully keep out the experimental measurements from our considerations. There is, however, a way to include them in an optimal way. This is done by our method [12] which will be presented in the following section. In the meantime this method has also been used by other authors (see e.g. Ref. [13]). The method is based on the fact that we can use global duality when the weight function is non-singular. This is the case for a polynomial function. Therefore, we might mimic the weight function by a polynomial function obeying different conditions which I will explain later. By adding and subtracting this polynomial function $P_N(s)$ of given order N to the weight function $H(s)$, without any restrictions we obtain

$$\int_{s_a}^{s_b} \rho(s)H(s)ds = \int_{s_a}^{s_b} \rho(s)\left(H(s)-P_N(s)\right)ds + \int_{s_a}^{s_b} \rho(s)P_N(s)ds \tag{17}$$

where $[s_a, s_b]$ is any interval out of the total integration range. But because the second term has now a polynomial weight, we can use global duality to write

$$\int_{s_a}^{s_b} \rho(s)P_N(s)ds \;=\; \frac{1}{2\pi i}\int_{s_a}^{s_b} Disc\,\Pi(s)P_N(s)ds \;=$$

$$= -\frac{1}{2\pi i}\oint_{|s|=s_a}\Pi(-s)P_N(s)ds + \frac{1}{2\pi i}\oint_{|s|=s_b}\Pi(-s)P_N(s)ds. \quad (18)$$

Thus this part can be represented by a difference of two circle integrals in the complex plane. On the other hand, the difference $H(s) - P_N(s)$ suppresses the contribution of the first part. Our method consists thus of the following steps:

- replacing $\rho(s)$ in the first part of Eq. (17) by the value of the experimentally measured total cross section $R(s)$ (see e.g. Ref. [1]),
- replacing the circle integral contribution to flavours at their threshold by zero,
- in all other cases inserting the QCD perturbative and non-perturbative parts of $\Pi(-s)$ on the circle.

These replacements can be seen as a concept within QCD sum rules. To obtain the best efficience of our method, we have to restrict the polynomial function by the following contraints:

- The method of least squares should be used to mimic the weight
- However, the degree N should not be higher than the order of the highest perturbative resp. non-perturbative contribution increased by one (this is a consequence of the Cauchy's theorem which is involved in the analytical integration of the circle integrals).
- Especially for the low energy region, the polynomial function should vanish on the real axis to avoid instanton effects.
- In regions where resonances occur, the polynomial function should fit the weight function to suppress those contributions which constitute the highest uncertainty of the experimental data.

What are the ingredients for this mixture then? We have to look at what experimental data we can use and how the perturbative expansions for the correlator function are given.

The experiment side

Data from experiments are crucial in the low energy range and in threshold regions where perturbative QCD cannot be applied. Here combined data sets from various electron positron annihilation experiments are used [1] which are accomplished by recent BES measurements [14].

In addition the use of precise τ-decay data from Ref. [15] by isospin rotation promises to be a rewarding step in the low energy region. The vector τ spectral functions are related to the isovector e^+e^- cross sections for the corresponding hadronic states X by [15]

$$\sigma^{I=1}(e^+e^- \longrightarrow X^0) = \frac{4\pi\alpha^2}{s}v_{J=1}(\tau^- \to X^-\nu_\tau). \quad (19)$$

$v_{J=1}(\tau^- \to X^-\nu_\tau)$ is obtained by dividing the normalized invariant mass-squared distribution $dN_{X^-}/N_{X^-}ds$ for a given hadronic mass $\sqrt{s}$ by the appropriate kinematic factor,

$$\begin{aligned} v_{J=1}(\tau^- \to X^-\nu_\tau) &= \frac{m_\tau^2}{6|V_{ud}|^2 S_{\rm EW}} \frac{B(\tau^- \to X^-\nu_\tau)}{B(\tau^- \to e^-\bar{\nu}_e\nu_\tau)} \times \\ &\times \frac{dN_{X^-}}{N_{X^-}ds}\left[\left(1-\frac{s}{m_\tau^2}\right)^2\left(1+\frac{2s}{m_\tau^2}\right)\right]^{-1} \end{aligned} \tag{20}$$

where $|V_{ud}| = 0.9752 \pm 0.0007$ denotes the CKM weak mixing matrix element [16] and $S_{EW} = 1 + \delta_{EW} = 1.0194$ accounts for electroweak second order corrections [17]. The spectral functions are normalized by the ratio of the respective vector branching fraction $B(\tau^- \to X^-\nu_\tau)$ to the branching fraction of the electron channel $B(\tau^- \to e^-\bar{\nu}_e\nu_\tau) = 17.79 \pm 0.04$ [19].

The theory side

The two-point correlator [18] is given by

$$i\int\langle 0|j_\alpha^{\rm em}(x)j_\beta^{\rm em}(0)|0\rangle e^{iqx}d^4x = (-g_{\alpha\beta}q^2 + q_\alpha q_\beta)\Pi(q^2) \tag{21}$$

where we only included the isospin contribution $I = 1$, in contrast to corresponding considerations for the τ decay. The scalar correlator function $\Pi(q^2)$ consists of perturbative and non-perturbative contributions which we include to the extend we need them to keep the accuracy. For the perturbative contribution to the correlator we use a result given in Ref. [20]. I only write down the first few terms,

$$\Pi^{\rm P}(q^2) = \frac{3}{16\pi^2}\sum_{i=1}^{n_f} Q_i^2\left[\frac{20}{9} + \frac{4}{3}L + C_F\left(\frac{55}{12} - 4\zeta(3) + L\right)\frac{\alpha_s}{\pi} + O(\alpha_s^2, m_q^2/q^2)\right] \tag{22}$$

with $L = \ln(\mu^2/q^2)$ while in Ref. [20] the expression is given up to the order $O(\alpha_s^2, m_q^{12}/q^{12})$. The number of active flavours is denoted by n_f. For the zeroth order term in the m_q^2/q^2 expansion we have added higher order terms in α_s,

$$\frac{3}{16\pi^2}\sum_{i=1}^{n_f} Q_i^2\left[\left(c_3 + k_2 L + \frac{1}{2}(k_0\beta_1 + 2k_1\beta_0)L^2 + \frac{1}{3}k_0\beta_0^2 L^3\right)\left(\frac{\alpha_s}{\pi}\right)^3 + O(\alpha_s^4)\right] \tag{23}$$

with $k_0 = 1$, $k_1 = 1.63982$ and $k_2 = 6.37101$. We have denoted the yet unknown constant term in the four-loop contribution by c_3. Remark, however, that the constant non-logarithmic terms will not contribute to our calculations. The non-perturbative contributions are given in ref. [21],

$$\begin{aligned} \Pi^{\rm NP}(q^2) &= \frac{1}{18q^4}\left(1 + \frac{7\alpha_s}{6\pi}\right)\langle\frac{\alpha_s}{\pi}G^2\rangle + \\ &+\frac{8}{9q^4}\left(1 + \frac{\alpha_s}{4\pi}C_F + \dots\right)\langle m_u\bar{u}u\rangle + \frac{2}{9q^4}\left(1 + \frac{\alpha_s}{4\pi}C_F + \dots\right)\langle m_d\bar{d}d\rangle + \\ &+\frac{2}{9q^4}\left(1 + \frac{\alpha_s}{4\pi}C_F + (5.8 + 0.92L)\frac{\alpha_s^2}{\pi^2}\right)\langle m_s\bar{s}s\rangle + \\ &+\frac{\alpha_s^2}{9\pi^2q^4}(0.6 + 0.333L)\langle m_u\bar{u}u + m_d\bar{d}d\rangle + \\ &-\frac{C_A m_s^4}{36\pi^2q^4}\left(1 + 2L + (0.7 + 7.333L + 4L^2)\frac{\alpha_s}{\pi}\right) - \frac{448\pi}{243q^6}\alpha_s|\langle\bar{q}q\rangle|^2 + O(q^{-8}) \end{aligned} \tag{24}$$

where we have included the m_s^4/q^4-contribution arising from the unit operator. In this expression we used the $SU(3)$ colour factors $C_F = 4/3$, $C_A = 3$, and $T_F = 1/2$. For the coupling constant α_s as well as for the running quark mass we use four-loop expression given in Refs. [22, 23, 24] even though in both cases the three-loop accuracy would already have been sufficient for the present application.

The evaluation of the QED coupling

The calculation for the running fine structure constant or QED coupling is published in Ref. [12]. An important step in preparing for the application of our method is to select points s_a and s_b for the limits of the integrals resp. the radii of the circles. Except for the threshold regions there seems to be a wide range for placing these points. Indeed we have shown that the method is fairly independent of this choice but is in some cases limited by computational constraints as for instance the fact that matrices fail to be inverted in some special cases. For this reason it is not advisable to make the intervals too narrow.

As an example for a threshold case I show the first contribution,

$$\int_{s_0}^{s_1} R(s)H(s)ds = \int_{s_0}^{s_1} R^{\rm exp}(s)\left(H(s) - P_N(s)\right)ds + 6\pi i \oint_{|s|=s_1} \Pi^{\rm QCD}(-s)P_N(s)ds. \quad (25)$$

Here we selected the range from the light flavour production threshold $s_0 = 4m_\pi^2$ going to the next threshold marked by the mass of the ψ, $s_1 = m_\psi^2 \approx (3.1\,{\rm GeV})^2$. In this case of course the inner circle integral did not contribute. The polynomial weight functions are compared to the weight function itself in Fig. 1, the results for different polynomial degrees are shown in Fig. 2. Further subdivisions can be found in Tab. 1. The part of the integral starting from $s_4 = (40\,{\rm GeV})^2$ up to infinity is done using local duality, i.e. by inserting the function $R(s)$ obtained for perturbative QCD,

$$\rho^{\rm had}(s) = \frac{\alpha N_c}{12\pi}\sum_f Q_f^2\sqrt{1 - \frac{4m_f^2}{s}}\left(1 + \frac{2m_f^2}{s}\right) \quad (26)$$

(N_c is the number of colours) into the second part of Eq. (17).

To obtain the results shown in Tab. 1, we used the condensate values

$$\langle\frac{\alpha_s}{\pi}GG\rangle = 0.04 \pm 0.04\,{\rm GeV}^4, \qquad \alpha_s\langle\bar qq\rangle^2 = (4 \pm 4)\times 10^{-4}\,{\rm GeV}^6. \quad (27)$$

For the errors coming from the uncertainty of the QCD scale we take

$$\Lambda_{\overline{\rm MS}} = 380 \pm 60\,{\rm MeV} \quad (28)$$

The errors resulting from the uncertainty in the QCD scale in different energy intervals are clearly correlated and will have to be added linearly in the end. We also include the error of the strange quark mass in the light quark region which is taken as

$$\bar m_s(1\,{\rm GeV}) = 200 \pm 60\,{\rm MeV} \quad (29)$$

For the charm and bottom quark masses we use the values

$$\bar m_c(m_c) = 1.4 \pm 0.2\,{\rm GeV}, \quad \bar m_b(m_b) = 4.8 \pm 0.3\,{\rm GeV}. \quad (30)$$

interval for $\sqrt{s}$	N	data contr.	contribution to $\Delta\alpha^{(5)}_{\rm had}(M_Z)$	error due to $\Lambda_{\overline{\rm MS}}$
$[0.28\,{\rm GeV}, 3.1\,{\rm GeV}]$	$1, 2$	24%	$(73.9 \pm 1.1) \times 10^{-4}$	0.9×10^{-4}
$[3.1\,{\rm GeV}, 9.46\,{\rm GeV}]$	$3, 4$	0.3%	$(69.5 \pm 3.0) \times 10^{-4}$	1.4×10^{-4}
$[9.46\,{\rm GeV}, 30\,{\rm GeV}]$	$3, 4$	1.1%	$(71.6 \pm 0.5) \times 10^{-4}$	0.06×10^{-4}
$[30\,{\rm GeV}, 40\,{\rm GeV}]$	$3, 4$	0.15%	$(19.93 \pm 0.01) \times 10^{-4}$	0.02×10^{-4}
$\sqrt{s} > 40\,{\rm GeV}$			$(42.67 \pm 0.09) \times 10^{-4}$	
total range			$(277.6 \pm 3.2) \times 10^{-4}$	1.67×10^{-4}

Table 1: Contributions of different energy intervals to $\alpha^{(5)}_{\rm had}(M_Z)$. Second column: choice of neighbouring pairs of the polynomial degree N. Third column: fraction of the contribution of experimental data [1]. Fourth column: contribution to $\Delta\alpha^{(5)}_{\rm had}(M_Z)$ with all errors included except for the systematic error due to the dependence on $\Lambda_{\overline{\rm MS}}$ which is separately listed in the fifth column.

Summing up the contributions from the five flavours u, d, s, c, and b, our result for the hadronic contribution to the dispersion integral including the systematic error due to the dependence on $\Lambda_{\overline{\rm MS}}$ (column 5 in Table 1) reads

$$\Delta\alpha^{(5)}_{\rm had}(M_Z) = (277.6 \pm 4.1) \times 10^{-4}. \tag{31}$$

In order to obtain the total result for $\alpha(M_Z)$, we have to add the lepton and top contributions. Since we have nothing new to add to the calculation of these contributions we simply take the values from Ref. [25],

$$\Delta\alpha^{t}_{\rm had}(M_Z) = (-0.70 \pm 0.05) \times 10^{-4}, \qquad \Delta\alpha_{\rm lep}(M_Z) \approx 314.97 \times 10^{-4}. \tag{32}$$

Writing $\Delta\alpha(M_Z) = \Delta\alpha_{\rm lep}(M_Z) + \Delta\alpha_{\rm had}(M_Z)$ our final result is

$$\alpha(M_Z)^{-1} = \alpha(0)^{-1}(1 - \Delta\alpha(M_Z)) = 128.925 \pm 0.056 \tag{33}$$

where we used $\alpha(0)^{-1} = 137.036$.

The evaluation of the magnetic moment

The procedure we choose for the anomalous magnetic moment of the muon differs from the one used for the QED coupling because of two reasons. As mentioned earlier, the singularity of the weight function at the origin on the one hand is much stronger and it therefore it would need more effort to approximate the weight by a polynomial in the lower energy region. On the other hand we can and should therefore make use of the fact that the τ data between $s = (0.28\,{\rm GeV})^2$ and $s_1 = (1.4\,{\rm GeV})^2$ were measured with a high precision, so they should influence the evaluation of the dispersion integral at full extent. The less precise e^+e^- data from the above energy region, however, are worth being replaced by QCD expressions as much as possible.

Our results are presented in Table 2. The dealing with error estimates is at some point different from the precious case of the QED coupling because of the different data sets we used. We only keep error estimates steming from the uncertainties for the quark masses

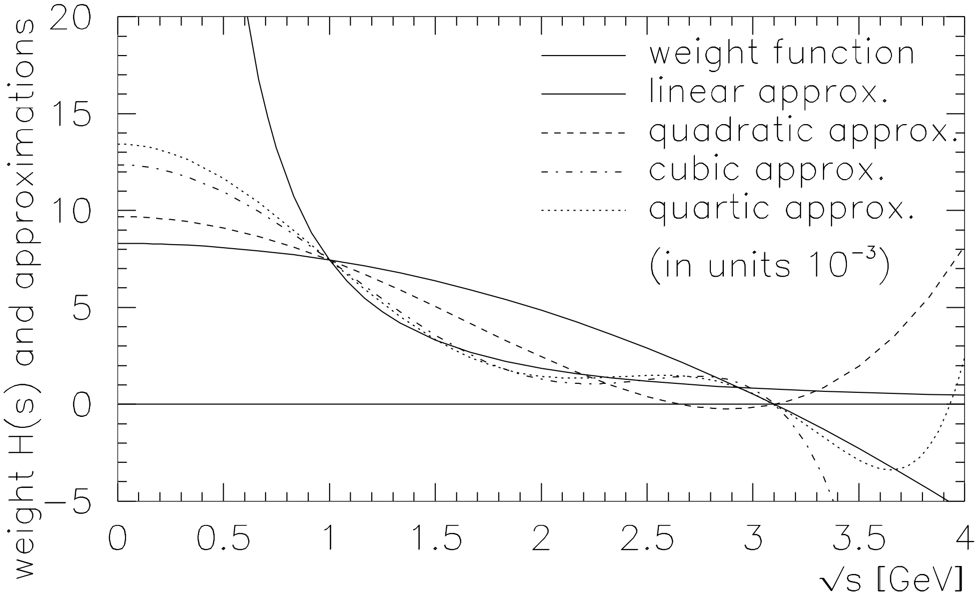


Figure 1: Weight function $H(s)$ and polynomial approximations $P_N(s)$ in the lowest energy interval $2m_\pi \leq \sqrt{s} \leq 3.1\,\mathrm{GeV}$.

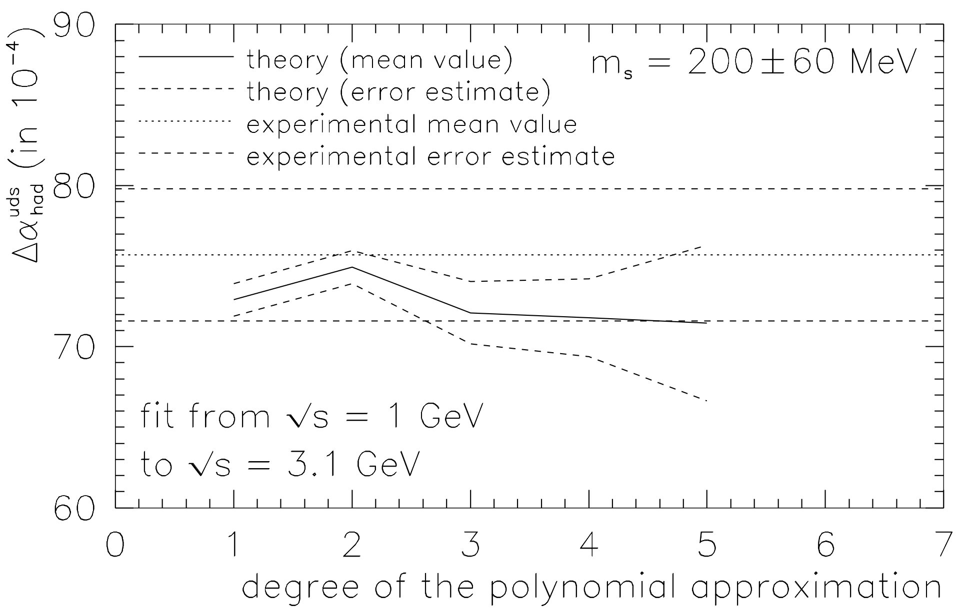


Figure 2: Comparison of the l.h.s. and r.h.s. of the sum rule given by Eq. (25) in the interval $0.28\,\mathrm{GeV} \leq \sqrt{s} \leq 3.1\,\mathrm{GeV}$. Dotted horizontal line: value of integrating the l.h.s. using experimental data including error bars [1]. The points give the values of the r.h.s. integration for various orders N of the polynomial approximation. Straight line interpolations between the points are for illustration only. The dashed lines indicate the error estimate of our calculation.

interval for $\sqrt{s}$	contributions to $a_\mu^{\rm had}$	comments
$[0.28\,{\rm GeV}, 1.4\,{\rm GeV}]$	$(530.4 \pm 6.1) \times 10^{-10}$	τ decay data
ω resonance	$(38.89 \pm 1.30) \times 10^{-10}$	e^+e^- annihilation data
ϕ resonances	$(40.37 \pm 1.17) \times 10^{-10}$	e^+e^- annihilation data
$[1.4\,{\rm GeV}, 3.1\,{\rm GeV}]$	$(52.13 \pm 2.04) \times 10^{-10}$	polynomial method
J/ψ resonances	$(8.81 \pm 0.61) \times 10^{-10}$	e^+e^- annihilation data
$[3.1\,{\rm GeV}, 40\,{\rm GeV}]$	$(22.13 \pm 1.14) \times 10^{-10}$	e^+e^- annihilation data
Υ resonances	$(1.21 \pm 0.19) \times 10^{-10}$	e^+e^- annihilation data
$[40\,{\rm GeV}, \infty]$	0.15×10^{-10}	theory
top quark contr.	$< 10^{-13}$	theory
whole range	$\pm 1.83 \times 10^{-10}$	uncertainty from $\Lambda_{\overline{\rm MS}}$
hadronic contr.	$(694.1 \pm 7.0) \times 10^{-10}$	

Table 2: The different contributions to the hadronic part of the anomalous magnetic moment $a_\mu^{\rm had}$ of the muon.

and a systematic error over all ranges from the uncertainty of the parameter $\Lambda_{\overline{\rm MS}}$. In comparison to this, all other error estimates for the theory contribution of the "mixed" regions (like errors of vacuum expectation values) can be neglected. The value obtained is comparable with the predictions in Ref. [13] and can therefore be seen as one of the most accurate on this field. It will be published hopefully soon [26].

4 Estimates for the next-to-leading order

At this point we come back to the point whether the experimental measurements for the hadronic e^+e^- channel really have to be taken into account. The question is to be taken serious because in this case it is at least strange to denote the estimate "theory estimate". The method to use experimental data for a theory estimate is different from the (in this respect more preferable) method to use the experimental data in order to extract a few parameters like the locations of resonances and their widths. In order to see whether such an approach is possible, we use simple models to extract a single parameter from the given value of the anomalous magnetic moment of the muon. On the first sight this looks like a "zero-sum game", but using the model we then can estimate the next-to-leading order contribution to the anomalous magnetic moment which is found to be very close to data-based results in the literature.

Integration in the Euclidean domain

As mentioned earlier, the non-analytic structure of the complex plane does not allow for the calculation of the spectral density (given in Minkowskian domain) by using perturbation theory results (obtained in Euclidean domain). Therefore, at this point we return to the Euclidean domain. The transition from

$$a_\mu = 4\pi^2 \left(\frac{\alpha}{\pi}\right)^2 \int_{4m_\pi^2}^\infty \frac{K(s)}{s}\rho(s)ds, \qquad K(s) = \int_0^1 \frac{z^2(1-z)dz}{z^2+(1-z)s/m^2} \tag{34}$$

to the integral representation

$$a_\mu = 4\pi^2 \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty W(t) \left(-\Pi^{\rm had}(t)\right) dt \tag{35}$$

where

$$W(t) = \frac{4m^4}{\sqrt{t^2+4m^2t}\left(t+2m^2+\sqrt{t^2+4m^2t}\right)^2} \tag{36}$$

is calculated in Appendix B. Finally, an integration-by-parts leads to

$$a_\mu = 4\pi^2 \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty \left(-\frac{d\Pi^{\rm had}(t)}{dt}\right) F(t)dt \tag{37}$$

where

$$F(t) = \int_t^\infty W(t')dt' = \frac{1}{2}\left(\frac{t+2m^2-\sqrt{t^2+4m^2t}}{t+2m^2+\sqrt{t^2+4m^2t}}\right) = \frac{2m^4}{\left(t+2m^2+\sqrt{t^2+4m^2t}\right)^2}. \tag{38}$$

For high values of the Euclidean squared energy t the expression $-d\Pi^{\rm had}(-t)/dt$ can be taken from perturbation theory while for the whole range different models can be used which are introduced in the following. However, we can take a short look on the weight function $F(t)$. This function shown in Fig. 3 for $m = m_\mu = (769.9 \pm 0.8)\,\mathrm{MeV}$ [16] supports to very high extend low energy contributions. This will become important in the following.

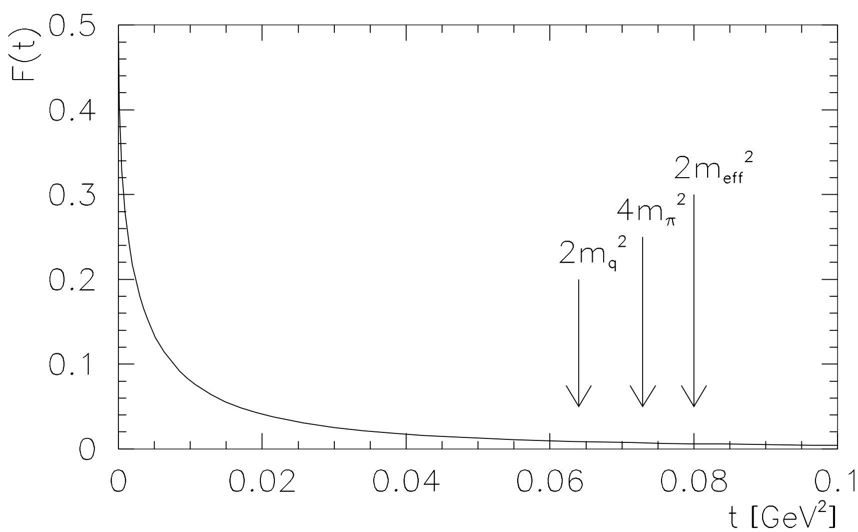


Figure 3: The LO Euclidean weight function $F(t)$

The step model

The simplest model we can find for the e^+e^- hadronic spectrum is the model of steps which appear when a new quark flavour is opened, i.e. when quark-antiquark pair production becomes kinematically possible at this specific energy. The model would simply read

$$\rho_1(s) = N_c Q^2 \theta(s - 4m_Q^2) \tag{39}$$

where $N_c = 3$ is the number of colours and Q is the charge (in units of the elementary charge) of the quark flavour with mass m_Q. Actually we need not separate between the three lightest flavours but instead can take them in combination,

$$\rho_1^{uds} = N_c(Q_u^2 + Q_d^2 + Q_s^2)\theta(s - 4m_{\rm eff}^2) = N_c Q_{\rm eff}^2 \theta(s - 4m_{\rm eff}^2), \qquad Q_{\rm eff}^2 = \frac{2}{3} \tag{40}$$

where $m_{\rm eff}$ is the mass parameter which has to be adjusted in this model. For the step functions of the heavy quarks we use their masses from literature as cited before. Starting with this model spectral function, we obtain for the hadronic correlator function

$$\Pi_1^{uds}(t) = -t \int_{4m_Q^2}^{\infty} \frac{\rho_1(s)}{s(s+t)} ds = -N_c Q_{\rm eff}^2 t \int_{4m_{\rm eff}^2}^{\infty} \frac{ds}{s(s+t)} = N_c Q_{\rm eff}^2 \ln\left(\frac{4m_{\rm eff}^2}{t + 4m_{\rm eff}^2}\right) \tag{41}$$

and for it's derivative

$$-\frac{d\Pi_1^{uds}(t)}{dt} = \frac{N_c Q_{\rm eff}^2}{t + 4m_Q^2}. \tag{42}$$

This actually makes sense, since perturbative QCD for high predicts

$$-\frac{d\Pi^{\rm had}(t)}{dt} = \frac{N_c Q_{\rm eff}^2}{t}\left(1 + \frac{\alpha_s(-t)}{\pi}\right) \tag{43}$$

While for small values of t this expression remains finite.

The threshold model

One can of course use more sophisticated models. One of these is given by the fermionic spectral density

$$\rho_2(s) = N_c Q_{\rm eff}^2 \sqrt{1 - \frac{4m_q^2}{s}}\left(1 + \frac{2m_q^2}{s}\right). \tag{44}$$

The corresponding hadronic correlator function is given by

$$\Pi_2(t) = N_c Q_{\rm eff}^2 \left\{\left(\frac{1}{z} - 3\right)\varphi(z) - \frac{1}{3}\right\} \tag{45}$$

where

$$\varphi(z) = \frac{1}{\sqrt{z}} \operatorname{artanh}(\sqrt{z}) - 1, \qquad z = \frac{t}{t + 4m_q^2}, \tag{46}$$

and

$$-\frac{d\Pi_2(t)}{dt} = N_c Q_{\rm eff}^2 \left\{\frac{t - 6m_q^2}{t^2} + \frac{24m_q^4}{t^3}\sqrt{\frac{t}{t + 4m_q^2}} \operatorname{artanh}\left(\sqrt{\frac{t}{t + 4m_q^2}}\right)\right\}. \tag{47}$$

It figures out that the derivatives of $\Pi_1^{uds}(t)$ and $\Pi_2(t)$ coincide at the origin if we chose $m_{\rm eff}/m_q = \sqrt{5}/2$.

The resonance model

Finally, one can also include the lowest lying resonance, namely the ρ meson resonance. A one-scale no-parameter model that satisfies the duality constraints from the operator product expansion is given by [27, 28, 29, 30]

$$\rho_4(s) = N_c Q_{\rm eff}^2 \left(2m_\rho^2 \delta(s-m_\rho^2) + \theta(s-2m_\rho^2)\right). \tag{48}$$

The derivative of the corresponding correlator function reads

$$-\frac{d\Pi_4(t)}{dt} = N_c Q_{\rm eff}^2 \left(\frac{2m_\rho^2}{(t+m_\rho^2)^2} + \frac{1}{t+2m_\rho^2}\right). \tag{49}$$

Alternatively, the narrow width ρ resonance can be replaced by a Breit–Wigner resonance,

$$\rho_3(s) = N_c Q_{\rm eff}^2 \left(\theta(2m_\rho^2 - s)\rho_{\rm BW}(s)\theta(s-4m_\pi^2) + \theta(s-2m_\rho^2)\right) \tag{50}$$

where

$$\rho_{\rm BW}(s) = \frac{2m_\rho^2}{\pi}\left(\frac{\Gamma_\rho m_\rho}{(s-m_\rho^2+\Gamma_\rho^2/4)^2 + \Gamma_\rho^2 m_\rho^2}\right), \tag{51}$$

the derivative of the correlator function is given by

$$-\frac{d\Pi_3(t)}{dt} = N_c Q_{\rm eff}^2 \left(\int_{4m_\pi^2}^{2m_\rho^2} \frac{\rho_{\rm BW}(s)ds}{(s+t)^2} + \frac{1}{t+2m_\rho^2}\right). \tag{52}$$

The big surprize

The spectral densities we have used are quite different. A glance on Fig. 4 convinces us that at least the spectral density of the third model is quite different from the two others. However, calculating the derivative of the correlator function for the three models, we obtain the result shown on the top of Fig. 5. The two curves of model 1 and model 2 coincide at the origin because we chose the relation $m_{\rm eff}/m_q = \sqrt{5}/2$. We can even adjust $m_{\rm eff}/m_q$ (and, therefore, m_q) in order to obtain a conicidence with the third curve at the origin. The surprizing result is that despite the fact that the spectral densities in Fig. 4 look quite different, all curves for the correlator derivatives have nearly the same shape. But because the weight function $F(t)$ supports contributions at low values of t this means that the method is *independent of the choice for the model*! The procedure, therefore, is as follows: we take the first model as the easiest one to obtain our results. The contributions to the anomalous magnetic moment of the muon for the heavy quark flavours can be calculated to be $a_\mu^{\rm had}({\rm LO;heavy}) = (71 \pm 18) \times 10^{-11}$. The leading order result from the last chapter is reduced by this value to result in $a_\mu^{\rm had}({\rm LO;light}) = (6870 \pm 72) \times 10^{-11}$. This value is used to adjust $m_{\rm eff}$. We obtain [31]

$$m_{\rm eff} = (201.7 \pm 1.2)\,{\rm MeV}. \tag{53}$$

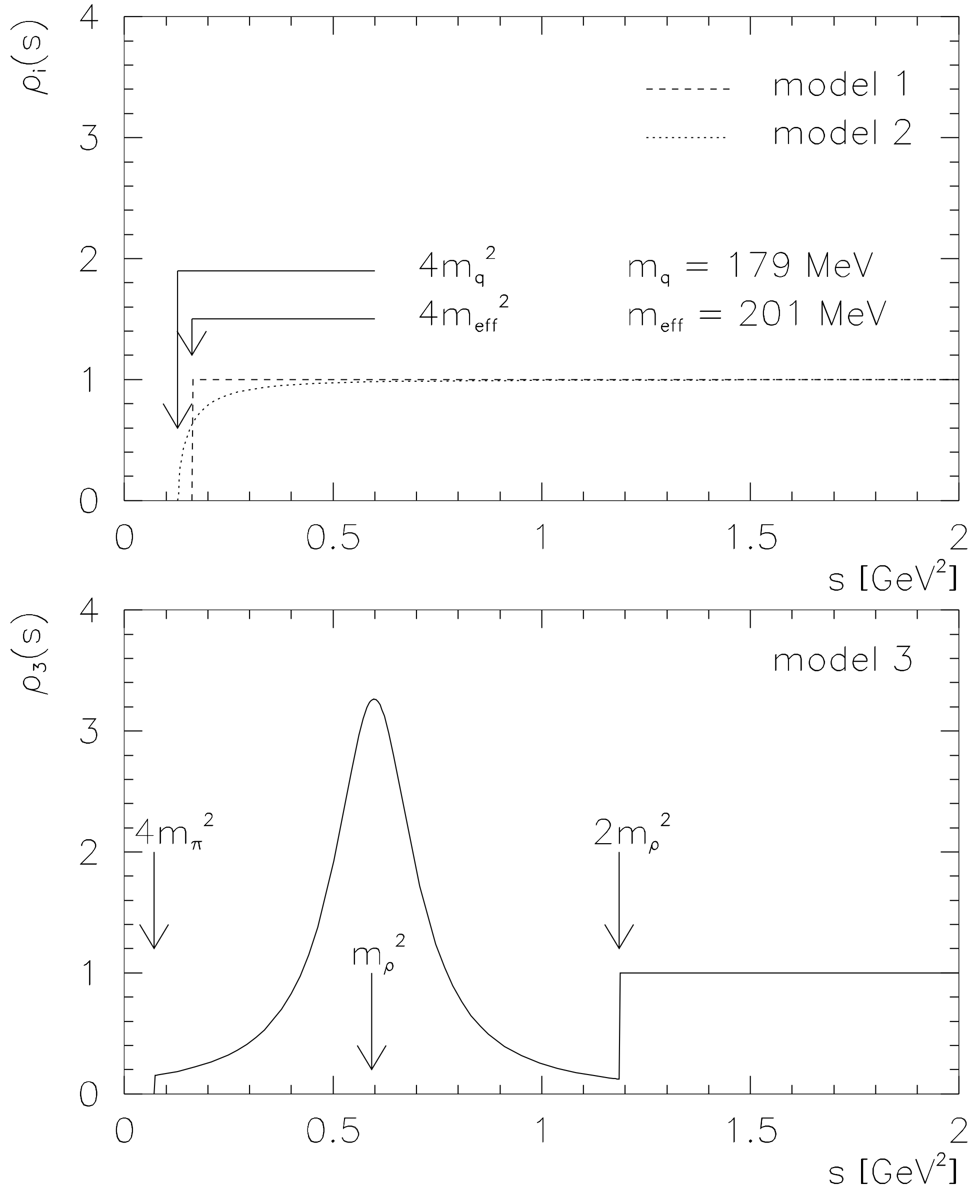
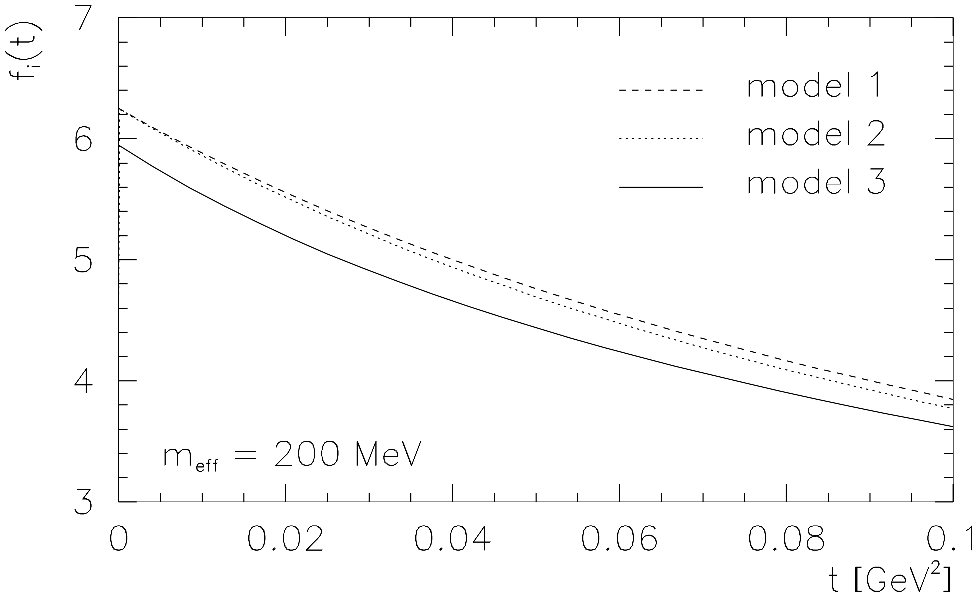


Figure 4: s-dependence of the spectral functions $\rho_1(s)$ of model 1 and $\rho_2(s)$ of model 2 (upper diagram), as compared to the spectral function $\rho_3(s)$ for model 3. We use $m_{\text{eff}} = 201\,\text{MeV}$, $m_q = 179\,\text{MeV}$, and the central values $m_\rho = 769.9\,\text{MeV}$ and $\Gamma_\rho = 150.2\,\text{MeV}$ [16].



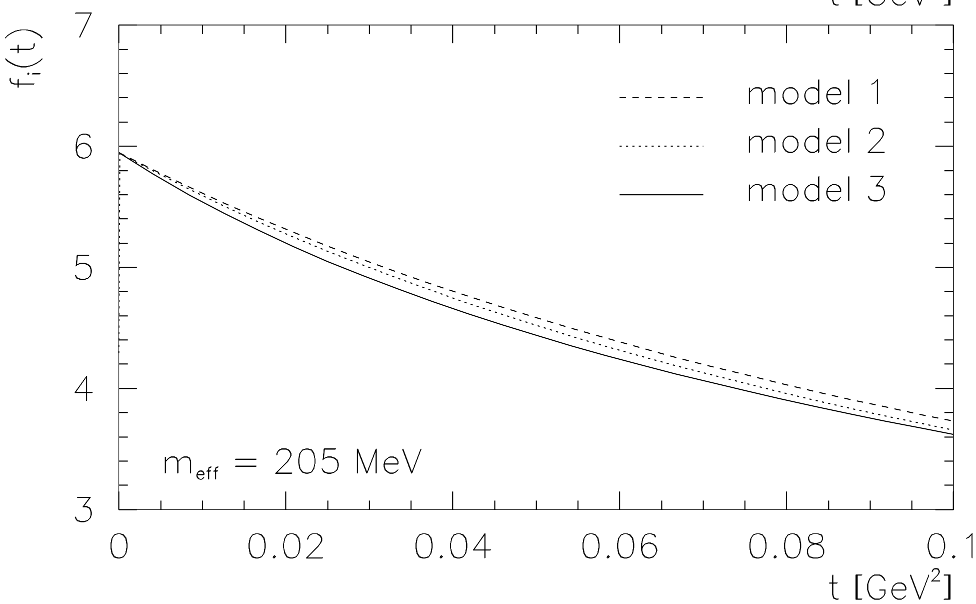


Figure 5: The functions $f_i(t)$ are the negative derivatives of the correlator function for the three different models. $m_{\text{eff}} = 200\,\text{MeV}$ is used for the upper diagram and $m_{\text{eff}} = 205\,\text{MeV}$ for the lower diagram. The parameter m_q used for the model 2 is connected to m_{eff} by $m_q = 2m_{\text{eff}}/\sqrt{5}$. The values $m_\rho = 769.9\,\text{MeV}$ and $\Gamma_\rho = 150.2\,\text{MeV}$ are taken from Ref. [16].

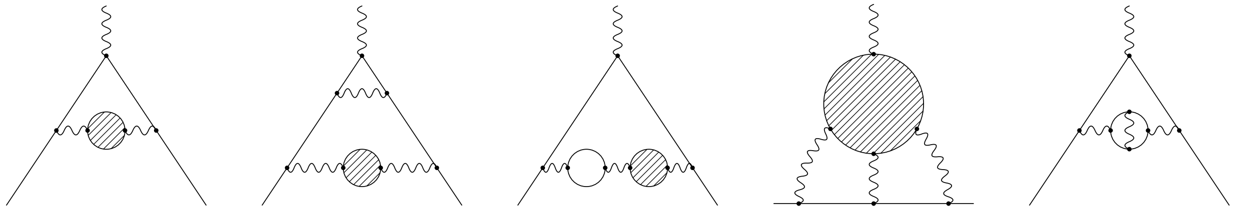


Figure 6: LO and NLO corrections to the anomalous magnetic moment

Next-to-leading order estimate

As I mentioned earlier, up to this point the method seems to be a "zero-sum game". However, there is really makes sense if we want to calculate the next-to-leading order correction. The correction shown in Fig. 6(b) is the convolution of the spectral density with the two-loop kernel $K^{(2)}(s)$,

$$a_\mu^{\rm had}({\rm NLO(b)}) = 4\pi\left(\frac{\alpha}{\pi}\right)^3\int_{4\pi^2}^\infty\frac{K^{(2)}(s)}{s}\rho(s)ds. \tag{54}$$

Using model 1, we obtain [31] $a_\mu^{\rm had}({\rm NLO(b)}) = (-211\pm 5)\times 10^{-11}$. For the two bubbles in Fig. 6(c), the so-called double-bubble diagram, model 1 can again employed to give [31] $a_\mu^{\rm had}({\rm NLO(c)}) = (106\pm 2)\times 10^{-11}$. Both results, as well as their sum are very close to the data-based results of the literature [32]. Finally, we add also the two next-to-leading order contributions, employing the four-point correlator, namely the light-by-light contribution (Fig. 6(d)) and the two photon Green function (Fig. 6(e)) (see the discussion in Ref. [31]) to obtain our estimate

$$a_\mu^{\rm had}({\rm NLO}) = (85\pm 20)\times 10^{-11}. \tag{55}$$

This estimate is still in good agreement with the difference

$$\begin{aligned} a_\mu^{\rm exp}({\rm NLO}) &= (7165\pm 151|_{\rm exp}\pm 2.9|_{\rm QED}\pm 4|_{\rm EW}-(6941\pm 70))\times 10^{-11} = \\ &= (224\pm 189)\times 10^{-11} \end{aligned} \tag{56}$$

between the experimental value and the theory estimate for the LO correction. The improvement of the accuracy of the experiment will show whether this can be kept or not.

5 Conclusion

I have presented results for the QED coupling α at the Z^0 pole and the anomalous magnetic moment a_μ of the muon, showing the main features of the polynomial adjustment method. In the second part I have used the LO theory estimate to adjust a mass parameter and calculate the NLO contributions.

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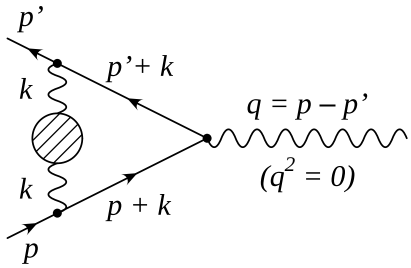
A How does the function $K(s)$ appear?



If we look at the figure at the left hand side which is the diagram we have to consider, the correlator (in Minkowskian domain) shown as shaded bubble is given by

$$\Pi_{\rho\sigma}(-k^2)=(k_\rho k_\sigma-k^2g_{\rho\sigma})\Pi(-k^2). \tag{57}$$

For the scalar correlator function we take

$$\Pi(-k^2)=\int\frac{k^2\rho(s)ds}{s(s-k^2)}. \tag{58}$$

The contribution for the diagram, therefore, is given by

$$\begin{aligned}
-ie\Lambda^\mu &= \int\frac{d^Dk}{(2\pi)^D}\bar u(p')(-ie\gamma^\alpha)\frac{i}{\not p'+\not k-m}(-ie\gamma^\mu)\frac{i}{\not p+\not k-m}(-ie\gamma^\beta)u(p)\times\\
&\times\frac{-i}{k^2}\left(g_{\alpha\rho}-(1-\chi)\frac{k_\alpha k_\rho}{k^2}\right)(-ik^2)\left(g^{\rho\sigma}-\frac{k^\rho k^\sigma}{k^2}\right)\Pi(-k^2)\frac{-i}{k^2}\left(g_{\sigma\beta}-(1-\chi)\frac{k_\sigma k_\beta}{k^2}\right)=\\
&= e^3\int\frac{d^Dk}{(2\pi)^D}\bar u(p')\frac{\gamma^\alpha(\not p'+\not k+m)\gamma^\mu(\not p+\not k+m)\gamma^\beta}{((p'+k)^2-m^2)((p+k)^2-m^2)k^2}\left(g_{\alpha\beta}-\frac{k_\alpha k_\beta}{k^2}\right)\Pi(-k^2)
\end{aligned} \tag{59}$$

while the anomalous magnetic moment is the part of the diagram proportional to $(p+p')^\mu$ for $q^2=(p-p')^2=0$. The contraction with the second part of the effective transverse photon propagator simplifies the expression remarkably. Because of

$$\begin{aligned}
\bar u(p')\not k(\not p'+\not k+m) &= \bar u(p')\left(2p'k+k^2+(-\not p'+m)\not k\right) = u(p')(2p'k+k^2),\\
(\not p+\not k+m)\not k &= \left(2pk+k^2+\not k(-\not p+m)\right)u(p) = (2pk+k^2)u(p)
\end{aligned} \tag{60}$$

(note that $\bar u(p')(\not p'-m)=(\not p-m)u(p)=0$), we obtain the two denominator factors

$$\begin{aligned}
(p'+k)^2-m^2 &= p'^2+2p'k+k^2-m^2 = 2p'k+k^2,\\
(p+k)^2-m^2 &= p^2+2pk+k^2-m^2 = 2pk+k^2
\end{aligned} \tag{61}$$

($p^2=p'^2=m^2$) which cancel out. However, the contribution figures out to be proportional to γ^μ and therefore does not contribute to the anomalous magnetic moment. Therefore, we can contract with $g_{\alpha\beta}$ and simplify the Dirac structure.

Simplification of the Dirac structure

The simplification is done systematically using $\bar u(p')(\not p'-m)=(\not p-m)u(p)=0$, $\gamma^\alpha\gamma_\alpha=D$ and the usual relations for the Dirac matrices,

$$\begin{aligned}
&\bar u(p')\gamma^\alpha(\not p'+\not k+m)\gamma^\mu(\not p+\not k+m)\gamma_\alpha u(p) =\\
&= \bar u(p')\left(2(p'+k)^\alpha+(-\not p'-\not k+m)\gamma^\alpha\right)\gamma^\mu\left(2(p+k)_\alpha+\gamma_\alpha(-\not p-\not k+m)\right)u(p) =\\
&= \bar u(p')\left(2(p'+k)^\alpha-\not k\gamma^\alpha\right)\gamma^\mu\left(2(p+k)_\alpha-\gamma_\alpha\not k\right)u(p) =
\end{aligned}$$

$$
\begin{aligned}
&= 4(pp' + (p+p')k + k^2)\bar{u}(p')\gamma^\mu u(p) + \bar{u}(p')\not{k}\gamma^\alpha\gamma^\mu\gamma_\alpha\not{k}u(p) + \\
&\quad -2\bar{u}(p')\gamma^\mu(\not{p}' + \not{k})\not{k}u(p) - 2\bar{u}(p')\not{k}(\not{p} + \not{k})\gamma^\mu u(p) = \\
&= (4m^2 + 4(p+p')k + 4k^2)\bar{u}(p')\gamma^\mu u(p) + (2-D)\bar{u}(p')\not{k}\gamma^\mu\not{k}u(p) + \\
&\quad -4p'^\mu\bar{u}(p')\not{k}u(p) + 2m\bar{u}(p')\gamma^\mu\not{k}u(p) - 2k^2\bar{u}(p')\gamma^\mu u(p) + \\
&\quad -4p^\mu\bar{u}(p')\not{k}u(p) + 2m\bar{u}(p')\not{k}\gamma^\mu u(p) - 2k^2\bar{u}(p')\gamma^\mu u(p) = \dots \\
&= \bar{u}(p')\Big((4m^2 + 4(p+p')k + (D-2)k^2)\gamma^\mu + \\
&\qquad -(4(p+p')^\mu + 2(D-2)k^\mu)\not{k} + 4mk^\mu\Big)u(p) \qquad (62)
\end{aligned}
$$

where $0 = q^2 = (p-p')^2 = 2m^2 - 2pp' \ \Rightarrow pp' = m^2$ was used. Again, the first part is proportional to γ^μ and does not contribute to the anomalous magnetic moment.

Scalar, vector and tensor integrals

Inserting the dispersion relation for $\Pi(-k^2)$, the denominator factor k^2 is replaced by $(s - k^2)$. The basic integral to deal with will be calculated by using Feynman parametrization,

$$
\begin{aligned}
I &= \int \frac{d^Dk}{(2\pi)^D} \frac{1}{(k^2 + 2pk)(k^2 + 2p'k)(k^2 - s)} = \\
&= \int \frac{d^Dk}{(2\pi)^D}\Gamma(3)\int_0^1\int_0^{1-z_1} \frac{dz_1dz_2}{(z_1(k^2+2pk) + z_2(k^2+2p'k) + (1-z_1-z_2)(k^2-s))^3} = \\
&= \Gamma(3)\int_0^1 dz_1\int_0^{1-z_1} dz_2 \int \frac{d^Dk}{(2\pi)^D}\frac{1}{(k^2 + 2(z_1p + z_2p')k - (1-z_1-z_2)s)^3} = \\
&=: \Gamma(3)\int_0^1 dz_1\int_0^{1-z_1} dz_2 \int \frac{d^Dk}{(2\pi)^D}\frac{1}{(k^2 + 2Pk - M^2)^3} = \\
&= \Gamma(3)\int_0^1 dz_1\int_0^{1-z_1} dz_2 \int \frac{d^Dk}{(2\pi)^D}\frac{1}{((k+P)^2 - P^2 - M^2)^3} = \\
&= \Gamma(3)\int_0^1 dz_1\int_0^{1-z_1} dz_2 \int \frac{d^Dk'}{(2\pi)^D}\frac{1}{(k'^2 - P^2 - M^2)^3} = \\
&= \frac{-i\Gamma(3-D/2)}{(4\pi)^{D/2}}\int_0^1 dz_1\int_0^{1-z_1} dz_2(P^2+M^2)^{D/2-3} \qquad (63)
\end{aligned}
$$

where for the last step a standard integral was used which is calculated by employing the Wick rotation. Note in addition that a shift $k \to k' = k + P$ was performed. If we start with a vector integral including a factor k^μ in the numerator, one obtains

$$
\begin{aligned}
I^\mu &= \Gamma(3)\int_0^1 dz_1\int_0^{1-z_1} dz_2 \int \frac{d^Dk'}{(2\pi)^D}\frac{(k'-P)^\mu}{(k'^2 - P^2 - M^2)^3} = \\
&= \frac{i\Gamma(3-D/2)}{(4\pi)^{D/2}}\int_0^1 dz_1\int_0^{1-z_1} dz_2 P^\mu(P^2+M^2)^{D/2-3}, \qquad (64)
\end{aligned}
$$

the integrand proportional to k'^μ vanishes because the function is odd in k'^μ. For the tensor integral we obtain

$$
\tilde{I}^{\mu\nu} = \frac{-i\Gamma(3-D/2)}{(4\pi)^{D/2}}\int_0^1 dz_1\int_0^{1-z_1} dz_2 P^\mu P^\nu(P^2+M^2)^{D/2-3} + g^{\mu\nu}J =: I^{\mu\nu} + g^{\mu\nu}J. \quad (65)
$$

Combining both ingredients

If we combine the Dirac structure and the integrals, we obtain (up to general factors)

$$-4(p+p')^\mu \gamma_\lambda I^\lambda - 2(D-2)\gamma_\rho I^{\mu\rho} - 2(D-2)\Gamma^\mu J + 4mI^\mu. \tag{66}$$

Again, the term proportional to J does not contribute, we do not have to care about this integral in the following. Including all general factors, we obtain

$$\begin{aligned} \Lambda^\mu &= \frac{e^2\Gamma(3-D/2)}{(4\pi)^{D/2}} \int_0^1 dz_1 \int_0^{1-z_1} dz_2 \times \\ &\quad \times \bar{u}(p')\left(4(p+p')^\mu \not{P} - 2(D-2)P^\mu \not{P} - 4mP^\mu\right) u(p)(P^2+M^2)^{D/2-3}. \end{aligned} \tag{67}$$

The integral is finite. Therefore, we can replace $D=4$. Now we use

$$\bar{u}(p')\not{P}u(p) = \bar{u}(p')(z_1\not{p} + z_2\not{p}')u(p) = m(z_1+z_2)\bar{u}(p')u(p) \tag{68}$$

and write $M = (1-z_1-z_2)s$ and

$$P^\mu = z_1 p^\mu + z_2 p'^\mu = \frac{1}{2}(z_1+z_2)(p+p')^\mu + \frac{1}{2}(z_1-z_2)(p-p')^\mu \tag{69}$$

where again only the first part counts for the anomalous magnetic moment. Finally, we can use $z = z_1 + z_2$ as a new variable and obtain

$$\begin{aligned} \Lambda^\mu &= \frac{e^2}{16\pi^2}\bar{u}(p')(p+p')^\mu u(p) \int_0^1 dz_1 \int_{z_1}^1 dz \frac{4mz - 2mz^2 - 2mz}{m^2z^2 + (1-z)s} = \\ &= \frac{e^2}{8\pi^2 m}\bar{u}(p')(p+p')^\mu u(p) \int_0^1 dz \int_0^z dz_1 \frac{z(1-z)}{z^2+(1-z)s/m^2} = \\ &= \frac{\alpha}{2\pi m}\bar{u}(p')(p+p')^\mu u(p) \int_0^1 \frac{z^2(1-z)dz}{z^2+(1-z)s/m^2}. \end{aligned} \tag{70}$$

At this point the function $K(s)$ appears.

Explicit form of $K(s)$

The integral for $K(s)$ can be calculated. First we can write

$$\begin{aligned} K(s) &= \int_0^1 \frac{z^2(1-z)dz}{z^2+(1-z)s/m^2} = \int_0^1 \frac{-z(z^2+(1-z)s/m^2) + (1-z)zs/m^2 + z^2}{z^2+(1-z)s/m^2} = \\ \ldots &= -\int_0^1 z\,dz + \left(1 - \frac{s}{m^2}\right)\int_0^1 dz + \int_0^1 \frac{zs/m^2(2-s/m^2) - s/m^2(1-s/m^2)}{z^2+(1-z)s/m^2} dz. \end{aligned} \tag{71}$$

The denominator in the integrand can be written as product

$$z^2 + (1-z)\frac{s}{m^2} = (z-z_+)(z-z_-) \tag{72}$$

where

$$z_\pm = \frac{s}{2m^2}(1\pm v) = \frac{2}{1\mp v}, \qquad v = \sqrt{1-\frac{4m^2}{s}}. \tag{73}$$

Therefore, we make the ansatz

$$\frac{A_+}{z-z_+}+\frac{A_-}{z-z_-}=\frac{(A_++A_-)z-(A_+z_-+A_-z_+)}{(z-z_+)(z-z_-)} \tag{74}$$

and solve the system of equations

$$A_++A_-=\frac{s}{m^2}\left(2-\frac{s}{m^2}\right),\qquad A_+z_-+A_-z_+=\frac{s}{m^2}\left(1-\frac{s}{m^2}\right) \tag{75}$$

to obtain finally

$$vA_+=-\frac{(1+v)^2}{(1-v)^2},\qquad vA_-=+\frac{(1-v)^2}{(1+v)^2}. \tag{76}$$

Using

$$1-z_\pm=1-\frac{2}{1\mp v}=-\frac{1\pm v}{1\mp v}\quad\Rightarrow\quad\frac{1-z_\pm}{-z_\pm}=\frac{1\pm v}{2}, \tag{77}$$

we obtain

$$\begin{aligned} K(s) &= \frac{1}{2}-\frac{4}{1-v^2}+A_+\ln\left(\frac{1-z_+}{-z_+}\right)+A_-\ln\left(\frac{1-v_-}{-v_-}\right) = \\ &= \frac{1}{2}-\frac{4}{1-v^2}-\frac{(1+v)^2}{v(1-v)^2}\ln\left(\frac{1+v}{2}\right)+\frac{(1-v)^2}{v(1+v)^2}\ln\left(\frac{1-v}{2}\right). \end{aligned} \tag{78}$$

In using $x=(1-v)/(1+v)$, we finally obtain

$$\begin{aligned} K(s) &= \frac{1}{2}-\frac{(1+x)^2}{x}-\frac{1+x}{x^2(1-x)}\ln\left(\frac{1}{1+x}\right)+x^2\frac{1+x}{1-x}\ln\left(\frac{x}{1+x}\right) = \\ &= \frac{1}{2}-\frac{(1+x)^2}{x}+\frac{1+x}{1-x}\left(\frac{1}{x^2}-x^2\right)\ln(1+x)+x^2\frac{1+x}{1-x}\ln x \end{aligned} \tag{79}$$

which can be shown to be identical with the expression in the main text.

B Calculation details for the three models

The main point here is to calculate the Euclidean analogon to the function $K(s)$, i.e. $W(t)$. We start with a reformulation of the denominator in the integrand of $K(s)$,

$$z^2+(1-z)\frac{s}{m^2}=\frac{1-z}{m^2}\left(s+\frac{z^2}{1-z}m^2\right) \tag{80}$$

and conclude that

$$t=\frac{z^2}{1-z}m^2\quad\Rightarrow\quad z=\frac{1}{2m^2}\left(-t+\sqrt{t^2+4m^2t}\right) \tag{81}$$

is the substitution we have to use. We obtain

$$\begin{aligned} dz &= \frac{1}{2m^2}\left(-1+\frac{t+2m^2}{\sqrt{t^2+4m^2t}}\right)dt = \frac{t+2m^2-\sqrt{t^2+4m^2t}}{2m^2\sqrt{t^2+4m^2t}}dt, \\ z^2 &= \frac{1}{4m^4}\left(t^2+t^2+4m^2t-2t\sqrt{t^2+4m^2t}\right) = \frac{t}{2m^4}\left(t+2m^2-\sqrt{t^2+4m^2t}\right), \\ 1-z &= \frac{1}{2m^2}\left(t+2m^2-\sqrt{t^2+4m^2t}\right) \end{aligned} \tag{82}$$

and are lucky that a general factor occurs at all places. Moreover, for this general factor we can use

$$\left(t+2m^2-\sqrt{t^2+4m^2t}\right)\left(t+2m^2+\sqrt{t^2+4m^2t}\right)=4m^4 \tag{83}$$

to switch freely from numerator to denominator and vice versa. For the denominator we obtain

$$z^2+(1-z)\frac{s}{m^2}=\frac{1}{2m^4}\left(t+2m^2-\sqrt{t^2+4m^2t}\right)(s+t), \tag{84}$$

for the numerator we get

$$z^2(1-z)dz=\frac{t(t+2m^2-\sqrt{t^2+4m^2t})^3}{8m^8\sqrt{t^2+4m^2t}}dt, \tag{85}$$

therefore, finally

$$\begin{aligned} K(s) &= \int_0^1 \frac{z^2(1-z)dz}{z^2+(1-z)s/m^2} = \int_0^\infty \frac{t(t+2m^2-\sqrt{t^2+4m^2t})^2}{4m^4\sqrt{t^2+4m^2t}(s+t)}dt = \\ &= \int_0^\infty \frac{4m^4t\,dt}{\sqrt{t^2+4m^2t}(t+2m^2+\sqrt{t^2+4m^2t})^2(s+t)} = \int_0^\infty \frac{tW(t)dt}{s+t}. \end{aligned} \tag{86}$$

Finally, we have

$$\begin{aligned} a_\mu &= 4\pi^2\left(\frac{\alpha}{\pi}\right)^2\int_{4m_\pi^2}^\infty \frac{K(s)}{s}\rho(s)ds = 4\pi^2\left(\frac{\alpha}{\pi}\right)^2\int_{4m_\pi^2}^\infty\int_0^\infty \frac{tW(t)}{s(s+t)}\rho(s)dt\,ds = \\ &= 4\pi^2\left(\frac{\alpha}{\pi}\right)^2\int_0^\infty W(t)\int_{4m_\pi^2}^\infty \frac{t\rho(s)}{s(s+t)}ds\,dt = 4\pi^2\left(\frac{\alpha}{\pi}\right)^2\int_0^\infty W(t)\left(-\Pi(t)\right)dt. \end{aligned} \tag{87}$$

The Euclidean weight $F(t)$

$W(t)$ was not the final weight function we selected for our considerations. Instead, we performed an integration-by-parts, taking the derivative of $\Pi(t)$ and integrating $W(t)$. The Euclidean weight function $F(t)$ should be given by

$$F(t)=\int_t^\infty W(t')dt'. \tag{88}$$

We show this by checking that $F(\infty)=0$ and $F'(t)=-W(t)$ for the fiven

$$F(t)=\frac{1}{2}\left(\frac{t+2m^2-\sqrt{t^2+4m^2t}}{t+2m^2+\sqrt{t^2+4m^2t}}\right)=\frac{2m^4}{(t+2m^2+\sqrt{t^2+4m^2t})^2}. \tag{89}$$

The first condition is granted. For the second one we take

$$\begin{aligned} F'(t) &= \frac{-4m^4(1+(t+2m^2)/\sqrt{t^2+4m^2t})}{(t+2m^2+\sqrt{t^2+4m^2t})^3} = \frac{-4m^4(t+2m^2+\sqrt{t^2+4m^2t})}{\sqrt{t^2+4m^2t}(t+2m^2+\sqrt{t^2+4m^2t})^3} = \\ &= \frac{-4m^4}{\sqrt{t^2+4m^2t}(t+2m^2+\sqrt{t^2+4m^2t})^2} = -W(t). \end{aligned} \tag{90}$$

The correlator for model 1

The first calculation is quite easy and short,

$$\begin{aligned}
\frac{\Pi_1(t)}{N_c Q_{\rm eff}^2} &= -t\int_{4m_\pi^2}^\infty \frac{\rho_1(s)ds}{s(s+t)} = -t\int_{4m_\pi^2}^\infty \frac{\theta(s-4m_{\rm eff}^2)}{s(s+t)} = -t\int_{4m_{\rm eff}^2}^\infty \frac{ds}{s(s+t)} = \\
&= -\int_{4m_{\rm eff}^2}^\infty \left(\frac{ds}{s}-\frac{ds}{s+t}\right) = -\ln\left(\frac{s}{s+t}\right)\Bigg|_{4m_{\rm eff}^2}^\infty = \ln\left(\frac{4m_{\rm eff}^2}{t+4m_{\rm eff}^2}\right), \qquad (91)\\
-\frac{d\Pi_1(t)}{dt} &= \frac{N_c Q_{\rm eff}^2}{t+4m_{\rm eff}^2}. \qquad (92)
\end{aligned}$$

The correlator for model 2

Starting with

$$\rho(s) = N_c Q_{\rm eff}^2\sqrt{1-\frac{4m^2}{s}}\left(1+\frac{2m^2}{s}\right) = \frac{1}{2}N_c Q_{\rm eff}^2 v(3-v^2), \qquad v=\sqrt{1-\frac{4m^2}{s}}, \quad s=\frac{4m^2}{1-v^2} \qquad (93)$$

we obtain

$$\begin{aligned}
\frac{\Pi_2(t)}{N_c Q_{\rm eff}^2} &= -t\int_{4m^2}^\infty \frac{ds}{s(s+t)}\sqrt{1-\frac{4m^2}{s}}\left(1+\frac{2m^2}{s}\right) = -t\int_0^1 \frac{v^2(3-v^2)dv}{4m^2+t(1-v^2)} = \\
&= -t\int_0^1 \frac{v^2(3-v^2)dv}{4m^2+t-tv^2} = -\int_0^1 \frac{v^2(3-v^2)dv}{a^2-v^2}, \qquad a^2 = 1+\frac{4m^2}{t}. \qquad (94)
\end{aligned}$$

We continue with

$$\begin{aligned}
\frac{\Pi_2(t)}{N_c Q_{\rm eff}^2} &= -\int_0^1 \frac{v^2(3-v^2)dv}{a^2-v^2} = -\int_0^1 v^2 dv - (a^2-3)\int_0^1 dv + a^2(a^2-3)\int_0^1 \frac{dv}{a^2-v^2} = \\
&= -\frac{1}{3} - (a^2-3) + (a^2-3)\int_0^1 \frac{dv}{1-v^2/a^2} = -\frac{1}{3} + (a^2-3)\left(\text{artanh}\left(\frac{1}{a}\right)-1\right). \qquad (95)
\end{aligned}$$

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